EQUILIBRIUM OF A LIQUID FILM ON A ROTATING SPHERE

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The present paper concerns the evolution of the figure and the conditions of detachment of a thin liquid film wetting a solid sphere as functions of the angular velocity of rotation of the sphere. At zero angular velocity the film forms a uniform layer on the sphere. As the angular velocity increases the film is first "squashed" at the poles due to centrifugal and capillary forces. The film in the polar area then experiences rupture and slides toward the equator. With further increases in velocity the ring-shaped film layer at the equator detaches itself from the sphere. The evolution of the equilibrium figure of such a film differs markedly from the equilibrium figures of a homogenous rotating liquid mass acted on by surface tension forces as investigated by several authors [1]. Formally this is attributable to the presence of a boundary condition (the wetting angle) at the boundary between the film and wetted sphere.

1. Formulation of the problem. We shall consider the axisymmetric equilibrium figures of a thin liquid film which coats a rotating solid sphere. These figures correspond to "solid rotation" states, i.e. to states in which only the azimuthal component of velocity,

$$v_\varphi = \Omega (x^2 + y^2)^{1/2} \tag{1.1}$$

differs from zero.

We choose the coordinate system in such a way that the z-axis is directed along the axis of rotation.

The boundary conditions at the free surface of the liquid film are of the form [1]

$$\frac{1}{2}\rho\Omega^2(x^2+y^2)+p_0=\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{1.2}$$

Here Ω is the angular velocity of rotation of the coated sphere, ρ is the density of the liquid making up the film, σ is the coefficient of surface tension, R_1 and R_2 are the principal radii of curvature of the free surface of the film, and p_0 is a constant appearing in the expression for the pressure. In the case where the liquid film coats the entire surface of the solid sphere and where there is no external pressure, p_0 represents the pressure at the axis of rotation.

To determine the shape of the free surface of the film we need merely (by virtue of axial symmetry) find its equation in the meridional cross section, e.g. in the cross section $y = 0$. Substituting the expressions for the radii of curvature

$$\frac{1}{R_1}=-\frac{d}{dx}\left(\frac{z'}{\sqrt{1+z'^2}}\right), \qquad \frac{1}{R_2}=-\frac{z'}{x\sqrt{1+z'^2}} \tag{1.3}$$

into (1.2), we obtain the equation of the curve $z = z(x)$ which describes the shape of the meridional cross section of the film

$$\pm\left[\frac{d}{dx}\left(\frac{z'}{\sqrt{1+z'^2}}\right)+\frac{z'}{x\sqrt{1+z'^2}}\right]=-\frac{p_0}{\sigma}-\frac{1}{2}\rho\frac{\Omega^2}{\sigma}x^2 \tag{1.4}$$

Multiplying (1.4) by x and integrating, we obtain

$$\pm \frac{xz'}{\sqrt{1+z'^2}} = -\frac{p_0}{\sigma} x^2 - \frac{1}{8} \rho \frac{\Omega^2}{\sigma} x^4 + c \tag{1.5}$$

Here c is the integration constant. Let us introduce the following new variables and dimensionless angular velocity:

$$z = RZ, \quad x = RX, \quad \omega = (^1/_8\, \rho \Omega^2 R^3 \sigma^{-1})^{1/2} \tag{1.6}$$

where R is the radius of the sphere.
Equation (1.5) becomes

$$\pm xz' / \sqrt{1+z'^2} = c + sx^2 + \omega^2 x^4 \tag{1.7}$$

or (if we solve it for z'),

$$\mp z' = \frac{c + sx^2 + \omega^2 x^4}{\sqrt{x^2 - (c + sx^2 + \omega^2 x^4)^2}} \tag{1.8}$$

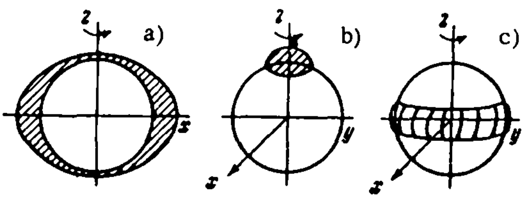

Fig. 1

In (1.7) and (1.8) we have preserved our earlier symbols for the coordinates and the constant c; instead of p_0 we have introduced the dimensionless parameter s.

In computing $z(x)$ we must take the minus sign in the first quadrant and the plus sign in the fourth quadrant in the left side of (1.8). To Eq. (1.8) we must add the condition of constant film volume and the wetting condition at the film-sphere boundary when such a boundary arises. These conditions enable us to determine all of the integration constants (including s and c).

We note that as long as the film wetting the sphere remains a connected domain its free surface can be zero-connected (shell, Fig. 1a), simply connected (drop, Fig. 1b), or doubly connected (belt, Fig. 1c).

2. Equilibrium of the shell. If the film covers the entire sphere, then its free surface is the same as in the case of a rotating homogeneous liquid mass. The rotation of a homogenous liquid mass is investigated in [1]. It is shown that with slow rotation it has the shape of an ellipsoid of rotation with the semiaxes $(1+g)$ along x and y and the semiaxis $(1+g)\,[1-\omega^2(1+g)^3]$ along z. In this case the quantity g can be found from the volume V of the film,

$$^4/_3\, \pi (1+g)^3\, [1 - \omega^2 (1+g)^3]\, R^3 = {}^4/_3\, \pi R^3 + V \tag{2.1}$$

Here and below we assume that the volume of the film is much smaller than that of the wetted sphere (that the film is thin). Here the film remains zero-connected (a shell) only in the case of slow rotations, i.e. for $g \ll 1$. Bearing this in mind we find from (2.1) that

$$g = {}^1/_4\, V / \pi R^3 + {}^1/_2\, \omega^2 \tag{2.2}$$

The thickness of the shell at the poles is given by

$$(1+g)\,[1-\omega^2(1+g)^2]-1={}^1/_4\,V/\pi R^3-{}^2/_3\,\omega^2 \tag{2.3}$$

Thus, with increasing angular velocity of rotation the thickness of the shell at the poles diminishes and finally becomes zero, when

$$\omega=\omega_1=({}^3/_8 V/\pi R^3)^{1/2} \tag{2.4}$$

Because of symmetry the shell ruptures at the two poles at the same time (provided its state is unperturbed). This results in the formation of a liquid belt with a doubly connected surface.

3. Equilibrium of the liquid belt. The state characterized by the liquid belt which replaces the shell state can, on the other hand, be considered quite independently of the shell state by virtue of the fact that a doubly connected film (belt) can also exist in the absence of rotation. The figure of this liquid belt, which is by hypothesis symmetric with respect to the equator, is described by Eq. (1.8). To this we must add the wetting condition

$$dz/dx=\operatorname{tg}(\alpha-\theta_*) \tag{3.1}$$

Here α is the angle of wetting (the contact angle); θ_* is the polar angle defining the position of the film-sphere boundary in the first quadrant. Because the belt is symmetric with respect to the equator we need only consider (1.8) in the first quadrant. The wetting condition at the second boundary of the belt lying in the fourth quadrant is then replaced by the condition that dz/dx become infinite at the equator, i.e. for $z=0$.

Let us consider the figure of the liquid belt for $\omega=0$. From (1.8) we have

$$-z=\int_{1+H}^{x}\frac{c+sx^2}{\sqrt{x^2-(c+sx^2)^2}}\,dx \tag{3.2}$$

We denote the dimensionless thickness of the belt for $z=0$ (at the equator) by H.

Integral (3.2) can be expressed in terms of elliptic integrals of the first and second kind. However, it is impossible to find the dependence of c and s on the wetting angle in explicit form. We shall therefore carry out our computations for the case $\alpha\approx{}^1/_2\,\pi$. The belt contracts to a narrow "cord" at the equator, and θ_* turns out to be close to ${}^1/_2\,\pi$. This enables us to simplify (3.2) by setting

$$x=1+u\qquad(u\ll 1) \tag{3.3}$$

Expanding the integrand in (3.2) in powers of u and retaining the principal terms only, we obtain

$$-z=\int_{H}^{u}\frac{(s+c)+u(s-c)}{\sqrt{1-[(s+c)+u(s-c)]^2}}\,du \tag{3.4}$$

Since $dz/du=-\infty$ for $u=H$, we have

$$(s+c)+H(s-c)=1 \tag{3.5}$$

Next, we integrate (3.4) with allowance for (3.5),

$$z^2 + \left(u + \frac{s+c}{s-c}\right)^2 = \frac{1}{(s-c)^2} \tag{3.6}$$

Thus, $z(x)$ is a circular arc of radius $1/(s-c)$ whose center can be placed on the surface of the wetted sphere $(s+c=0)$, inside the sphere $(s+c>0)$, or outside it $(s+c<0)$. From geometric considerations (for a thin film) we see that the wetting angle is close to $^1/_2\pi$ for $s+c=0$, that $\alpha < {}^1/_2\pi$ for $s+c>0$ and that $\alpha > {}^1/_2\pi$ for $s+c<0$. From (3.1) and (1.8) we infer that

$$-\sin(\alpha - \theta_*) = (s+c) + u_*(s-c) \tag{3.7}$$

The quantities θ_* and u_* can be found by recalling that they are the coordinates of the intersection of circle (3.6) with the circle $x^2 + z^2 = 1$. Computing the volume of the liquid belt with the same degree of accuracy as all our previous calculations (which enables us to use Guldin's theorem), we find that

$$\frac{V}{4\pi R^3} = \int_{x_*}^{1+H} z(x)\,dx - \int_{x_*}^{1} \sqrt{1-x^2}\,dx = (s-c)^{-2}[\sin(\alpha-\theta_*)\cos(\alpha-\theta_*) +$$
$$+ {}^1/_2\pi + \sin(\alpha-\theta_*)] + O(z_*^3) \tag{3.8}$$

Limiting ourselves to the case $\alpha \approx {}^1/_2\pi$, we begin by setting $\theta_* = \alpha$. We then find from (3.8) that

$$\frac{1}{s-c} = \left(\frac{V}{2\pi^2 R^3}\right)^{1/2} \tag{3.9}$$

From (3.6) we find that $z_* = 1/(s-c)$. Recalling that $z_* = \cos\theta_*$, we obtain

$$\theta_* = \alpha = \frac{\pi}{2} - \frac{1}{s-c} = \frac{\pi}{2} - \left(\frac{V}{2\pi^2 R^3}\right)^{1/2} \approx \frac{\pi}{2} \tag{3.10}$$

Finally, from (3.5) - (3.7) we have

$$\frac{s+c}{s-c} = \frac{1}{2(s-c)^2} = \frac{V}{4\pi^2 R^3}, \qquad H = \frac{1}{s-c} = \left(\frac{V}{2\pi^2 R^3}\right)^{1/2} \tag{3.11}$$

Similar computations carried out for $\alpha - \theta_* \neq 0$ yield θ_* as a function of α in the form

$$\theta_* = \frac{\pi}{2} - \left(\frac{V}{2\pi^2 R^3}\right)^{1/2}\left[1 + \frac{2}{\pi}\left(\frac{\pi}{2} - \alpha\right)\right] \tag{3.12}$$

We assume here that $^1/_2\pi - \alpha$ is small. Equation (3.12) shows that θ_* depends weakly on α, i.e. that over a large range of contact angles the quantity θ_* remains close to $^1/_2\pi$, and the liquid layer lies at the equator.

4. Detachment of the film. As ω increases the shape of the liquid belt changes, bulging out at the equator. For some value of the angular velocity an inflection point (Fig. 2a) appears in the cross section of the free surface. This corresponds to the case where the right side of (1.8) has a multiple root. Here

$$\frac{s^2}{4\omega^4} - \frac{c}{\omega^2} = 0 \quad (4.1)$$

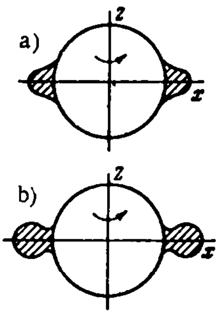

Fig. 2

Since the right side of (1.8) vanishes not more than twice (in the first quadrant), the inflection point clearly does not appear for wetting angles larger than $\frac{1}{2}\pi$, since for $\alpha > \frac{1}{2}\pi$ there is always an x such that $z' = 0$. This is a maximum point, and therefore cannot be an inflection point. Further on we shall consider wetting angles close to $\frac{1}{2}\pi$ but not exceeding $\frac{1}{2}\pi$. With such contact angles further increases in angular velocity result in a resolution of the inflection point into a maximum point which drifts away from the sphere and a minimum point lying close to the sphere. The film cross section resembles that of a bubble (with a neck at the minimum point) (Fig. 2b). With increasing ω the thickness of the neck diminishes and finally becomes zero (the film becomes detached from the sphere).

Let $\alpha < \frac{1}{2}\pi$. We denote the positions of the minimum and maximum of the function $z(x)$ by x_1 and x_2, respectively. Next, we introduce the notation

$$x_2 - x_1 = \Delta, \qquad x = x_1 + \Delta(\xi + \tfrac{1}{2}), \qquad A = 4\omega^2\Delta^2 \quad (4.2)$$

Recalling that $\Delta\xi \ll 1$ and retaining only the principal terms in (1.8), we obtain

$$z = -\int\limits_{\xi_0}^{\xi} \frac{\Delta A(\xi^2 - 1/4)\,d\xi}{[1 - A^2(\xi^2 - 1/4)^2]^{1/2}} \qquad \left(\xi_0 = \left(\frac{4+A}{4A}\right)^{1/2}\right) \quad (4.3)$$

Here ξ_0 is the value of ξ at $z = 0$ (i.e. at the equator). Let us convert to the new variable

$$\xi = [(4 + A)/4A]^{1/2}\cos t \quad (4.4)$$

in the integrand.

Equation (4.3) now becomes $(k^2 = 1/8\,(4 + A))$

$$z = \frac{\Delta}{\sqrt{2A}}\int\limits_0^t \frac{1 - 2k^2\sin^2 t}{\sqrt{1 - k^2\sin^2 t}}\,dt = \frac{\Delta}{\sqrt{2A}}\,[-F(t, k) + 2E(t, k)] \quad (4.5)$$

Here F and E are elliptic integrals of the first and second kind.

At the instant of detachment of the film the neck thickness, i.e. $z(x_1)$, becomes equal to zero,

$$\int\limits_0^{t_0} \frac{1 - 2k^2\sin^2 t}{\sqrt{1 - k^2\sin^2 t}}\,dt = 0 \quad (4.6)$$

Since the position of the neck corresponds to the condition $z' = 0$, this implies that

$$1 - 2k^2\sin^2 t_0 = 0 \quad (4.7)$$

Of the two roots of Eq. (4.7) the smaller root corresponds to the maximum of the function $z(x)$; the larger root corresponds to the minimum, i.e. to the neck. This large root is equal to

$$t_0 = \tau = \pi - \arcsin (1 / k\sqrt{2}) \tag{4.8}$$

Thus, condition (4.6) becomes

$$\int_0^{\tau} \frac{1 - 2k^2 \sin^2 t}{\sqrt{1 + k^2 \sin^2 t}} dt = 0 \tag{4.9}$$

Referring to tables of elliptic integrals [2], we can use this equation to find the value of k^2, and then use (4.4) to find the value of A for the state corresponding to detachment of the film,

$$k^2 = 0{,}73, \qquad A = 1.84 \tag{4.10}$$

The angular velocity for the detachment state can be readily determined in the case where the wetting angle is exactly $\frac{1}{2}\pi$. In this case the neck lies on the sphere, so that $\theta_* = \frac{1}{2}\pi$, and the film becomes completely detached.

Calculating the volume of the film whose figure is given by Formula (4.5) and equating the resulting volume to the specified volume V, we obtain

$$4\pi R^3 \, 0.272 \, \Delta^2 = V \tag{4.11}$$

From (4.2), (4.10), (4.11) we obtain the dimensionless angular velocity of detachment,

$$\omega = (\tfrac{1}{2} \pi R^3 / V)^{1/2} \tag{4.12}$$

Calculations carried out for $\alpha < \frac{1}{2}\pi$ (but $\frac{1}{2}\pi - \alpha \ll 1$) indicate that the angular velocity of detachment decreases with decreasing α .

$$\omega = \left[1 - \frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)^2\right]\left(\frac{\pi R^3}{2V}\right)^{1/2} \tag{4.13}$$

For $\alpha < \frac{1}{2}\pi$ part of the film remains on the sphere after detachment of the greater portion of the liquid mass.

In conclusion we note that random perturbations or deviations of the surface of the wetted body from sphericity can result in rupture of the film at one of the poles only at small angular velocities. Evolution of the film figure in this case takes the form of a liquid belt not symmetric with respect to the equator and requires further investigation.

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